Correspondences and quantum description of Aharonov–Bohm and Aharonov–Casher effects

2004 J. Phys. A: Math. Gen. 37 973

(http://iopscience.iop.org/0305-4470/37/3/029)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 973–988 PII: S0305-4470(04)61209-1

Correspondences and quantum description of Aharonov–Bohm and Aharonov–Casher effects

Minchul Lee[1] **and M Y Choi**[1,2]

[1] Department of Physics, Seoul National University, Seoul 151-747, Korea
[2] Korea Institute for Advanced Study, Seoul 130-722, Korea

Received 22 March 2003, in final form 17 September 2003
Published 7 January 2004
Online at stacks.iop.org/JPhysA/37/973 (DOI: 10.1088/0305-4470/37/3/029)

Abstract

We establish systematic consolidation of the Aharonov–Bohm (AB) and Aharonov–Casher effects including their scalar counterparts. Their formal correspondences in acquiring topological phases are revealed on the basis of the gauge symmetry in non-simply-connected spaces and the adiabatic condition for the state of magnetic dipoles. In addition, investigation of basic two-body interactions between an electric charge and a magnetic dipole clarifies their appropriate relative motions and discloses physical interrelations between the effects. Based on the two-body interaction, we also construct an exact microscopic description of the AB effect, where all the elements are treated on equal footing, i.e., magnetic dipoles are described quantum-mechanically and electromagnetic fields are quantized. This microscopic analysis not only confirms the conventional (semiclassical) results and the topological nature but also allows one to explore the fluctuation effects due to the precession of the magnetic dipoles with the adiabatic condition relaxed.

PACS numbers: 03.65.Vf, 03.75.Dg

1. Introduction

Since the celebrated work of Aharonov and Bohm [1], topological phases in quantum mechanical systems have been investigated intensively, both theoretically [1–16] and experimentally [17–19]. There have been identified four kinds of effects for quantum particles with electrical charges or magnetic moments, which give rise to non-trivial and experimentally observable topological phases: the Aharonov–Bohm (AB) effect and its scalar counterpart [1] (SAB) stand for accumulation of topological phases of electric charges moving, without ever being in electromagnetic fields, in vector and scalar potentials, respectively. On the other hand, development of topological phases by neutral particles with magnetic moments, which travel in electric and magnetic fields though experiencing no force, are addressed as the Aharonov–Casher (AC) [2] and the scalar AC (SAC) [3] effects, respectively. (In some of the literature, the latter is referred to as the SAB effect.)

Since the proposal, there have been many misleading attempts to deny the existence of topological phases in the AC and SAC effects, for example, possible classical interpretations [4, 5] and locality analysis of the AC effect [6–8]. Despite these, the AB and AC effects have been pointed out similar and even equivalent [9, 10] to each other. Furthermore, the apparent difference [11] between the AB and AC effects can be understood by addressing the underlying principle for the generation of topological phases in both cases [10]. It is thus tempting to extend such topological examination to their scalar counterparts, the SAB and SAC effects, and, further, to prove that these four complete all kinds of topological phase acquired by a system of an electric charge and a magnetic moment. Such investigation gives a clue for another interrelation between the effects, viewed in terms of the two-body interaction between an electric charge and a magnetic moment, one of which, being congregated macroscopically, constitutes a source of electromagnetic fields. This helps us deepen the interpretation for the SAB effect among others, which has not yet been considered with the source included, and manifest that all four effects simply arise from appropriate relative motions of electric charges and magnetic dipoles. With the sources taken into consideration, in particular, fully quantum-mechanical establishment of the effects can now be achieved by quantizing the electromagnetic fields; this is to be compared with the conventional (semiclassical) setup, where a (quantum) particle moves in *classical* electromagnetic fields. Indeed, a recent study of the AB effect, where sources for magnetic fields are treated quantum-mechanically, has yielded results almost the same as those in the conventional setup, although the perturbative nature of the analysis makes the conclusion short of rigorousness [12]. With rigorousness, this type of analysis would not only clarify hidden conditions for the degree of freedom in the magnetic moment, which has invoked controversial problems in the interpretation of the AC effect, but also make it possible to examine the possible effects of the precession of the magnetic moment.

In this paper, we establish systematic consolidation of the four topological effects. Inspecting the condition for gauge symmetry, we find a formal correspondence between the SAB and SAC effects, in addition to the known one between the AB and AC effects, and demonstrate their completeness in acquiring non-trivial topological phases. Also exposed is the importance of the adiabatic condition for magnetic dipoles in substantiation of the topological phase. Further, investigation of fundamental two-body interactions between constituent particles consisting of the total system reveals physical interrelations between the AC and SAB effects as well as between the AB and SAC effects. Here, the relative motions of the particles appropriate for the effects are classified. We next consider the quantum dynamics of the total system consisting of an electric charge and magnetic dipoles to construct an exact quantum description of the AB effect. By means of the path integral representation, we analyse the charge–dipole system interacting via photons, i.e., quantized fields; this confirms the AB effect obtained in the semiclassical setup and reveals the origin of the topological nature of the effect. Finally, we justify the adiabatic condition in real experiments and give a microscopic argument for the survival of the nonlocal and topological phase even in the presence of weak precession of the magnetic dipoles.

This paper is organized as follows: in section 2, we examine systematically the general condition for the formal equivalences among the effects, together with their completeness. In particular, the missing correspondence between the SAB and SAC effects is revealed. Section 3 is devoted to the physical interrelations between the effects. Presented in section 4 is the microscopic quantum description of the AB effect. Section 5 discusses the validity of the adiabatic condition and the effects of the precession of magnetic dipoles. Finally, the main results are summarized in section 6.

2. Formal correspondence between Aharonov–Bohm and Aharonov–Casher effects

The essential common property of AB and AC effects including their scalar counterparts is acquisition of topological and nonlocal phases in force-free regions. Such similarity indicates structural likeness in the equations of motion for the system and the key point for the topological phase is known to be the presence of gauge symmetry in non-simply-connected systems [10]. The same notion can be applied to obtain the correspondence between the SAB and SAC effects, allowing us to establish systematic consolidation of all four kinds of effect. In the following, we set up the condition for the gauge symmetry in the relativistic formulation.

Let us first look into the well-known AB and SAB effects. The Dirac equation for a spin-1/2 particle with charge q and mass m in external electromagnetic potential A_μ reads

$$\left(\gamma^\mu p_\mu - \frac{q}{c}\gamma^\mu A_\mu - mc\right)\psi = 0 \tag{1}$$

where ψ is the Dirac spinor, $p_\mu\,(\mu = 0, 1, 2, 3)$ is the covariant momentum operator, and γ^μ is the gamma matrix. Note that under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Gamma$, the gauge invariance principle requires the wavefunction ψ to transform as $\psi \to \psi' = \psi\exp[-(\mathrm{i}q/\hbar c)\Gamma]$ and consider a spacetime region where both the electric and magnetic fields vanish but the potential does not. In this region, from the relations $\mathbf{E} = -(1/c)(\partial\mathbf{A}/\partial t) - \nabla A_0 = 0$ and $\mathbf{B} = \nabla\times\mathbf{A} = 0$, one can choose a gauge Γ such that $A_\mu = -\partial_\mu\Gamma$, namely $\mathbf{A} = \nabla\Gamma$ and $A_0 = -(1/c)(\partial\Gamma/\partial t)$. This indicates that in the field-free region ($\mathbf{E} = \mathbf{B} = 0$) A_μ becomes a pure gauge and can be gauged away. The Dirac equation then transforms to one for a free particle:

$$(\gamma^\mu p_\mu - mc)\psi' = 0 \tag{2}$$

where the transformed wavefunction is given by

$$\psi' = \psi\exp\left[-\frac{\mathrm{i}q}{\hbar c}\Gamma\right] = \psi\exp\left[\frac{\mathrm{i}q}{\hbar c}\int A_\mu\,\mathrm{d}x^\mu\right]. \tag{3}$$

Suppose that the spacetime structure for the field-free region is non-simply connected and has in it an impenetrable region in which the field $\mathbf{E}$ or $\mathbf{B}$ persists. If an electric charge circulates around a closed path in the field-free region which encloses the impervious region, the wavefunction acquires a non-trivial and gauge-invariant topological phase

$$\phi_{\mathrm{TP}} \equiv \frac{q}{\hbar c}\oint A_\mu\,\mathrm{d}x^\mu = \frac{q}{\hbar c}\oint(A_0c\,\mathrm{d}t - \mathbf{A}\cdot\mathrm{d}\mathbf{x}). \tag{4}$$

In the AB setup (see figure 1(*a*)) with a time-independent magnetic field confined inside a shielded region, the acquired phase is given by $\phi_{\mathrm{TP}} = -(q/\hbar c)\oint\mathbf{A}\cdot\mathrm{d}\mathbf{x} = -(q/\hbar c)\int_S\mathbf{B}\cdot\mathrm{d}\mathbf{s} = -(q\Phi/\hbar c)$, where Φ is the magnetic flux passing through the area S enclosed by the path. On the other hand, in the SAB setup (see figure 1(*b*)), the charged particle moves in a time-dependent potential without ever being in the electric field, and one obtains the phase $\phi_{\mathrm{TP}} = (q/\hbar c)\oint A_0c\,\mathrm{d}t = (q/\hbar)\int\delta V(t)\,\mathrm{d}t$, where $\delta V(t)$, the potential difference at time t, arises from the non-zero electric field across the prohibited spacetime region.

Next the conditions for the AC and SAC effects are examined in a similar manner to the AB effect. Following the existing study of the AC effect, we begin with the Dirac equation for a neutral spin-1/2 particle with the magnetic dipole moment μ in the electromagnetic field $F^{\mu\nu}$:

$$\left(\gamma^\mu p_\mu - \frac{\mu}{2c}\sigma^{\mu\nu}F_{\mu\nu} - mc\right)\psi = 0 \tag{5}$$

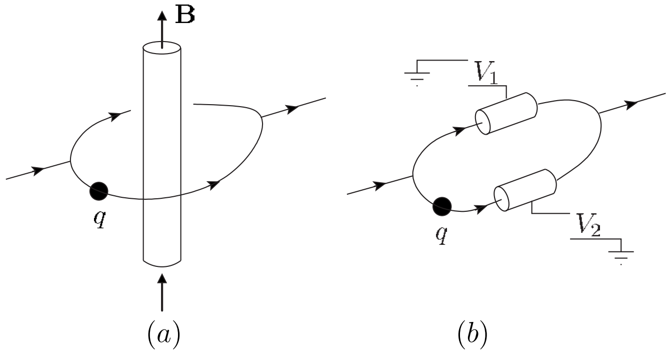

Figure 1. Schematic setups for (*a*) the AB and (*b*) the SAB effects.

with $\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]$, which can be written in the form

$$\left(\gamma^\mu p_\mu + \frac{\mathrm{i}\mu}{c}\gamma \cdot \mathbf{E}\gamma^0 + \frac{\mu}{c}\sigma \cdot \mathbf{B} - mc\right)\psi = 0 \tag{6}$$

with the Pauli matrix $\boldsymbol{\sigma}$. In contrast with equation (1) for the AB effect, the field itself enters into the equation of motion and cannot be simply gauged away, which implies that some restriction on the field should be imposed to make it behave like a pure gauge. The appropriate restriction may be uncovered by probing the condition that the equation of motion in equation (5) can be transformed to that for a free particle in a multiply-connected region, given by equations (2) and (3) with pure gauge potential A_μ. In general, one can write the potential in the form $A_\mu \equiv \tau\gamma^0 a_\mu$, where a_μ is a linear function of $\mathbf{E}$ and $\mathbf{B}$ and τ is a 4×4 matrix to be determined in the following. Comparison of equations (2) and (3) with equation (6) yields the relations:

$$[\tau, \gamma^0] = 0 \qquad \text{and} \qquad \{\tau, \gamma^k\} = 0 \qquad \text{if} \quad p_k\psi \neq 0 \quad (k = 1, 2, 3) \tag{7}$$

and

$$\mathrm{i}\mu\gamma \cdot \mathbf{E} = q\gamma \cdot \mathbf{a}\tau \qquad \text{and} \qquad \mu\sigma \cdot \mathbf{B} = -q\gamma^0\tau\gamma^0 a_0. \tag{8}$$

The matrix τ can be expressed as a linear combination of the complete set of 4×4 matrices: $1, \gamma^\mu, \sigma^{\mu\nu}, \gamma_5\,(\equiv \gamma^0\gamma^1\gamma^2\gamma^3)$ and $\gamma^\mu\gamma_5$. The commutation relations for matrix τ, given by equation (7), can be satisfied only provided at least one of $p_k\psi$ vanishes, which implies that the dimension for the AC effect is less than 3. In the case that only one of $p_k\phi$ is zero, say, $p_3\psi = 0$, one obtains a two-dimensional solution $\tau = \sigma^{12}$. This leads to the potential

$$q\mathbf{A} = -\mu\sigma_3(E_2, -E_1, 0) \qquad \text{and} \qquad qA_0 = -\mu\sigma_3 B_3 \tag{9}$$

with $E_3 = B_1 = B_2 = 0$. Requiring A_μ to behave like a pure gauge potential, we have $\nabla \times \mathbf{A} = 0$ and $-(1/c)(\partial\mathbf{A}/\partial t) - \nabla A_0 = 0$, which are followed by the charge-free condition $4\pi\rho = \nabla \cdot \mathbf{E} = 0$ and the current-free condition $(4\pi/c)\mathbf{J} = \nabla \times \mathbf{B} - (1/c)\partial\mathbf{E}/\partial t = 0$. On the other hand, when two of $p_k\psi$ are set equal to zero, for example, only $p_3\psi \neq 0$, we obtain a one-dimensional solution of the form $\tau = \mathrm{i}\gamma^3\gamma_5$, which again leads to equation (9) with minor sign flips and the same charge- and current-free conditions.

Hence, in order to develop a topological phase, a neutral spin-1/2 particle should have its path residing on a plane (for the AC effect) or on a straight line (for the SAC effect) in the multiply-connected region with neither charge nor current ($\rho = \mathbf{J} = 0$) and enclosing impenetrable regions with $\rho \neq 0$ or $\mathbf{J} \neq 0$. As in the AB effect, the non-trivial topological phase can be computed from equation (4) with A_μ given by equation (9). In the AC setup (see

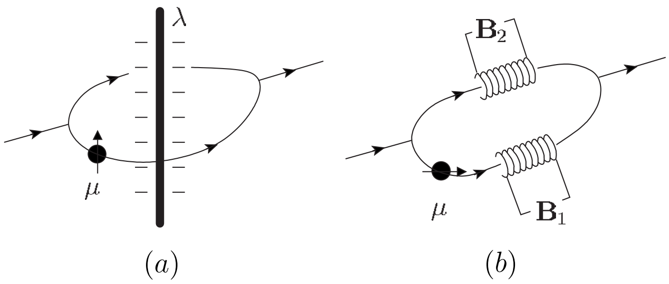

Figure 2. Schematic setups for (*a*) the AC and (*b*) the SAC effects.

Table 1. Properties of the multiply-connected and the impervious regions appropriate for the four kinds of topological effect, together with the characteristic dimension of the space holding the paths of the moving particle and the corresponding topological phase.

	Multiply-connected region	Impervious region	Dimension	Topological phase
AB	$\mathbf{E} = \mathbf{B} = 0$	$\mathbf{B} \neq 0$	Three	$q\Phi/\hbar c$
SAB	$\mathbf{E} = \mathbf{B} = 0$	$\mathbf{E} \neq 0$	Three	$(q/\hbar)\int \delta V(t)\,\mathrm{d}t$
AC	$\rho = \mathbf{J} = 0$	$\rho \neq 0$	Two	$4\pi\mu\lambda/\hbar c$
SAC	$\rho = \mathbf{J} = 0$	$\mathbf{J} \neq 0$	One	$(\mu/\hbar)\int \delta B(t)\,\mathrm{d}t$

figure 2(*a*)), where the magnetic dipole circulates on a plane around an infinitely long wire with charge line density λ, the topological phase is given by $\phi_{\mathrm{TP}} = 4\pi\mu\lambda/\hbar c$. On the other hand, in the SAC setup shown in figure 2(*b*), the magnetic dipole moves on a straight line under a time-varying magnetic field aligned along the direction of the dipole, and the phase is given by $\phi_{\mathrm{TP}} = (\mu/\hbar)\int \delta B(t)\,\mathrm{d}t$, where the magnetic-field difference $\delta B(t)$ arises from the non-zero current flowing inside the closed path.

Tabulated in table 1 are the conditions for the four kinds of topological effect as well as the corresponding topological phases. There the distinction between the AB and the AC effects is made clear: in the AB and SAB effects the conditions are described in terms of fields, whereas in the AC and SAC effects the sources such as charges and currents take the role of fields. In addition, one can infer from the completeness of the electromagnetic conditions listed in table 1 that these four kinds encompass all possible non-trivial interference phenomena for a system consisting of electric charges and magnetic dipoles.

It is straightforward to prove, under the conditions in table 1 and from equation (9) with A_μ interpreted as the potential in the AB setup, that the Dirac equations in equations (1) and (5) are exactly equal to each other, although in the case of the AC and SAC effects the form of the pseudo-potential A_μ is more severely constrained. This establishes exactly the formal correspondence between the SAB and SAC effects as well as that between the AB and AC effects proposed already [9], and manifests the key role of the gauge symmetry. Note, however, that the equivalence is possible only when one can replace the matrix σ_3 in equation (9) by one of its eigenvalues, +1 or −1. In fact the topological phase for the AC effect obtains the known form in table 1 only under this assumption. Since σ_3 is an operator in the spin space, such substitution implies that the spin of the magnetic dipole should remain in one of the eigenstates or more likely in the ground state during its motion along the path. Namely, the magnetic dipole moves so slowly that its state does not change, irrespective of its interaction with the field, which is the *adiabatic condition*. It is of interest that the adiabatic condition is also hidden in the AB and SAB effects: the magnetic or electric field necessary for the AB or SAB setup can be generated by a collection of stationary or moving magnetic dipoles, the details of which are discussed in section 3. In order to build up electromagnetic fields appropriate for the AB and SAB effects, the magnetic dipoles should be in their ground states at all times. Both the AB and the AC effects, therefore, are based on adiabatic processes, and it is concluded that the adiabatic condition as well as the gauge symmetry is essential for the topological phase.

We close this section with a comment about the force-free condition. In the AB and SAB setups the charge moves in the field-free region so that it does not feel any force. The force-free condition for the AC and SAC effects, where nonvanishing fields are present, has been proved classically in the spirit of total momentum conservation [5, 11]. Here we present a short sketch of the proof in the quantum version. We begin with the non-relativistic Hamiltonian with $\mathbf{B} = 0$:

$$\mathcal{H} = \frac{1}{2m}\left(\mathbf{p} - \frac{\mu}{c}\sigma \times \mathbf{E}\right)^2 - \frac{\mu^2}{2mc^2}\mathbf{E}^2 \tag{10}$$

with the charge- and current-free conditions $\nabla \cdot \mathbf{E} = \mathbf{J} = 0$. The Heisenberg equation of motion, with tedious applications of commutation relations, then yields

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle \mathbf{x}\rangle = \frac{\mu}{mc}\langle(\sigma \cdot \nabla)(\mathbf{E} \times \mathbf{p})\rangle + \frac{\mu^2}{m\hbar c^2}\langle[\mathbf{p} \times \mathbf{E}, \sigma \cdot \mathbf{E}]\rangle. \tag{11}$$

Note that with $E_z = p_z = 0$, the right-hand side has only the irrelevant z-component, which vanishes if the spin is in its ground state and $\mathbf{E}$ does not depend on z. We thus observe that the force-free condition in both the AB and the AC effects comes naturally from the gauge symmetry and the adiabatic condition.

3. Physical interrelation between Aharonov–Bohm and Aharonov–Casher effects

In section 2, we have examined the formal equivalence between the AB and AC effects. The *formal* equivalence stands for the agreement of the Dirac equation describing the effects via the one-to-one correspondence such as equation (9), without regard to the physical origin. Comparison between the electromagnetic conditions listed in table 1, however, gives us a clue for *physical* interrelations between the AB and AC effects, specifically the AB versus SAC effects and the AC versus SAB effects. To investigate such agreement, we consider a stationary or moving assembly of magnetic dipoles and electric charges as sources of electromagnetic fields for the AB and AC effects, respectively. Namely, stationary magnetic dipoles and electric charges generate the magnetic and electric fields, respectively, while moving ones induce the other fields. For the appropriate AB and AC setups, the assemblies are required to have specific structures and motions: the AB and SAB experiments consist of a single electric charge interacting with a structured assembly of magnetic dipoles. In the AC and SAC experiments the roles of the electric charge and the magnetic dipole are reversed, the details of which are explained below. Here the linearity of electrodynamics allows one to write the overall interaction as a sum of two-body interactions between an electric charge and a magnetic dipole. Accordingly, both effects are intrinsically based on the interaction between an electric charge and a neutral particle with magnetic moment; the same nature of the underlying interaction suggests that the effects are physically interrelated and equivalent to each other. Indeed we devise a setup for the SAB effect and newly find the physical interrelation between the AC and SAB effects. In the following analysis, the subscripts q and m are used to denote quantities pertaining to the charge and the magnetic dipole, respectively.

We first consider the AB and SAC effects, the interrelation between which has been demonstrated by Comay [13]. For completeness here we review briefly the argument. In the AB experiment, the region with non-zero magnetic field can be constructed by an infinitely long wire consisting of identical atoms or molecules with magnetic moment μ aligned along the wire. On the other hand, time-varying magnetic fields needed for the SAC effect can be associated with the motion of charges along the wire of a solenoid, which the magnetic dipole penetrates. From these configurations, we can identify the similar structure of the relative motion between the electric charge and the magnetic dipole, i.e., an electric charge circling around a (stationary) magnetic dipole. What is different between the two effects is that in the SAC setup the motion of the charge is confined on a plane and the magnetic dipole has another freedom of motion in the direction perpendicular to the plane. This freedom of motion for the magnetic dipole generates an electric field that acts on the charges in the solenoid, but the field does not affect the current in the solenoid; this may be easily proved by showing the line integral of the electric field along the path of the charge vanishes as long as the magnetic dipole points along its moving direction [13], so the motion of the dipole can be ignored. Hence the only relevant interaction in the SAC experiment is $\mu \cdot \mathbf{B}_q$, where $\mathbf{B}_q$ is the magnetic field induced by the moving charge, and in the AB experiment, $(q/c)\mathbf{v}_q \cdot \mathbf{A}_m$, where $\mathbf{v}_q$ and $\mathbf{A}_m$ are the velocity of the charge and the vector potential generated by the magnetic dipole, respectively. We now prove that these two interactions are equal to each other in the origin. Denoting $\mathbf{r} \equiv \mathbf{r}_q - \mathbf{r}_m$, where $\mathbf{r}_q$ and $\mathbf{r}_m$ are the positions of the charge and the magnetic dipole, respectively, we write the magnetic field induced by a moving charge and the vector potential of a magnetic dipole:

$$\mathbf{B}_q = \frac{\mathbf{v}_q}{c} \times \frac{q}{4\pi} \frac{(-\mathbf{r})}{r^3} \qquad \text{and} \qquad \mathbf{A}_m = \frac{1}{4\pi} \frac{\boldsymbol{\mu} \times \mathbf{r}}{r^3} \tag{12}$$

which are related via

$$\mu \cdot \mathbf{B}_q = \frac{q}{c}\mathbf{v}_q \cdot \left(\frac{1}{4\pi} \frac{\mu \times \mathbf{r}}{r^3} \right) = \frac{q}{c}\mathbf{v}_q \cdot \mathbf{A}_m. \tag{13}$$

Hence the two-body interactions in these experiments are intrinsically the same and the Schrödinger equations with the same interaction term govern the effects. The only difference between them is simply which phase change is to be measured: the electric charge in the AB effect and the magnetic dipole in the SAC effect.

A similar relation can be found between the AC and SAB effects. In the AC experiment, as illustrated in figure 2(a), the singular structure is constructed by an infinitely long straight wire of charges. On the other hand, the region of a time-dependent electric field in the SAB setup may be built up by an infinitely long hollow cylinder (i.e., cylindrical shell) of finite thickness, filled with identical magnetic dipoles aligned along the axis of the cylinder. Rotating the cylinder along its axis generates an electric field in the radial direction only inside the shell. This setup can be understood by modelling the magnetic moment as a current loop. Summing all the nearby current loops, one can find that azimuthal currents flow in the inner and outer surfaces of the cylindrical shell in opposite directions. Although in the frame of the cylinder charges distribute in such a way that no electric field exists, the rotation of the cylinder makes a difference between the charge densities on the two surfaces in the lab frame, which induces radial electric fields inside the shell, producing the potential difference between the two surfaces. In these two setups for the AC and SAB effects, another type of relative motion between an electric charge and a magnetic dipole is identified: a magnetic dipole circling around a (stationary) charge. Similar to the case of the SAC effect, there exists in the SAB effect the freedom of motion for the charged particle along the axis of the cylinder, which induces a magnetic field acting on the magnetic dipoles. Being in the azimuthal direction at all places, however, the magnetic field induced by the moving charge is perpendicular to the magnetic dipole, having no effect on the state of the dipole at all. Accordingly, the motion of the charge is irrelevant to our investigation, leaving the relevant two-body interactions $-(1/c)\mathbf{v}_m \cdot \mu \times \mathbf{E}_q$ in the AC experiment, where $\mathbf{v}_m$ and $\mathbf{E}_q$ are the velocity of the dipole and the electric field due to the charge, respectively, and qA_{0m} in the SAB experiment, where A_{0m} is the electric (scalar) potential induced by the moving magnetic dipole. To examine the relation between these two, we first note that the potential A_{0m}, given by the integration of the electric field $\mathbf{E}_m$ induced by the magnetic dipole, is thus related to the vector potential $\mathbf{A}_m$ generated by the magnetic dipole

$$A_{0m} = -\int \mathbf{E}_m \cdot \mathrm{d}\mathbf{l} = \int \frac{\mathbf{v}_m}{c} \times (\nabla \times \mathbf{A}_m) \cdot \mathrm{d}\mathbf{l} = \frac{\mathbf{v}_m}{c} \cdot \mathbf{A}_m. \tag{14}$$

On the other hand, the standard expressions for the electric field due to charge q and the vector potential by magnetic dipole μ lead to the relation

$$-\frac{\mathbf{v}_m}{c} \cdot \mu \times \mathbf{E}_q = \frac{q}{c}\mathbf{v}_m \cdot \frac{\mu \times \mathbf{r}}{4\pi r^3} = \frac{q}{c}\mathbf{v}_m \cdot \mathbf{A}_m \tag{15}$$

which, together with equation (14), shows

$$qA_{0m} = -\frac{\mathbf{v}_m}{c} \cdot \mu \times \mathbf{E}_q. \tag{16}$$

It is thus concluded that identical two-body interactions are responsible for both the AC and SAB effects and that these two effects are physically equivalent to each other.

Equation (13) indicates that the AB and SAC effects are identical in their physical origin whereas equation (16) discloses similar equivalence between the AC and SAB effects. The difference between the corresponding effects lies simply in which to observe: the phase acquire by the electric charge or that by the magnetic dipole. At this stage one may recall the proof that, based on the Galilean invariance, only the relative motion of the charged particle and the magnetic dipole enters into the Lagrangian of the total system and consequently, the AB and AC effects are the same in underlying physics [2]. There has also been an attempt to interrelate the AB/AC and SAB/SAC effects by changing the Lorentz frame from a stationary one to a co-moving one [14]. However, attention should be paid to the lack of consistent quantum treatment in those conventional analyses, where a quantum particle moves in a classical field, namely the source generating the field is treated classically and is immune to quantum fluctuations. For example, in the AB effect the electric charge is a quantum object while the magnetic dipoles as well as its field are regarded as classical, and vice versa in the AC effect. Thus in the conventional setup the difference between the AB and AC effects is not merely the reference frame, making it desirable to give a full quantum description.

Before closing this section, we comment on the quantum state of the magnetic dipole. One can see that throughout the above analysis the magnetic dipole is pinned to point in a specific direction: along the magnetic/charged wire in the AB/AC effect and along the axis of the solenoid/cylinder in the SAB/AC effect. This condition has also been used to argue for the irrelevance of the extra motion of the particle in the SAB/SAC effect. To achieve this condition in quantum mechanics, the quantum state of the dipole must remain in an eigenstate of the angular momentum in the supposed direction throughout the experiment. From this, we encounter again the adiabatic condition proposed in section 2, thus the adiabatic condition is also important to establishing the interrelation between those effects.

4. Full quantum description of Aharonov–Bohm effect

The AB and AC effects are inherently quantum phenomena: they describe the phase acquired by the wavefunction of a moving particle, either an electric charge (in the AB/SAB effect) or

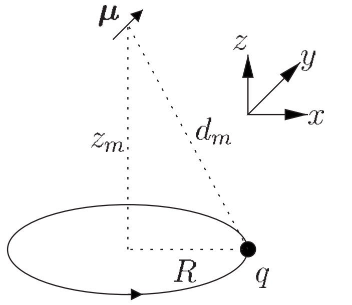

Figure 3. A particle with charge q moves around a circle on the xy-plane, in the presence of a fixed magnetic moment μ located on the z-axis, i.e., on the axis of the circle.

a magnetic dipole (in the AC/SAC effect), which cannot be dealt with by classical mechanics. Accordingly, the particle is treated as a quantum object, the motion of which is governed by the Schrödinger equation. Nonetheless, the conventional theory of the effects involves a semiclassical approximation in that the sources of fields and potentials are treated classically [12]. Although a source is a macroscopic object composed of a large number of particles (see section 3), the quantum state of each particle in the source is still important for substantiating the effect, as discussed in section 2. Moreover, it is not possible in the conventional semiclassical approach to study the effects of quantum fluctuations of the field and source. For a precise quantum interpretation, it is thus desirable to give a quantum description of the total system including the source, and in this section we focus on the exact quantum description of the AB effect, where magnetic dipoles as well as the moving electric charge are treated as quantum objects and the field is quantized, expressed as a linear combination of creation and annihilation operators of photons. The interaction between the charge and the magnet is then mediated via photons. Our aim here is to confirm the result of the conventional approach, to clarify the topological nature in the exact quantum description, and to lay a basis for studying the precession effects of the magnetic dipole, given in section 5.

As a preliminary, we first consider a system of a moving charged particle and a spatially fixed magnetic particle. The magnetic particle is in the eigenstate of the angular momentum J_u in the $\hat{\mathbf{u}}$-direction with eigenvalue $\hbar J$, so that it has a magnetic dipole moment given by $\mu = g\mu_B J\hat{\mathbf{u}}$, where μ_B is the Bohr magneton and g the Landé g-factor. In this system the particle, having charge q and mass m, is confined to move around a circle on the xy-plane whereas the magnet is located on the z-axis, i.e., on the axis of the circle, as illustrated in figure 3. Thus the distance between the particle and the magnetic moment is kept constant and given by $d_m = \sqrt{R^2 + z_m^2}$, where R is the radius of the circle and z_m is the distance of the magnetic dipole from the centre of the circle. The charged particle and the magnetic dipole interact with each other via virtual photon exchange and are described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\left[\mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{r})\right]^2 - \mu \cdot \mathbf{B}(\mathbf{r}_m) + \sum_{\mathbf{k},\hat{\epsilon}} \hbar\omega n_{\mathbf{k}\hat{\epsilon}} \qquad (17)$$

where $\mathbf{p}$ and $\mathbf{r}$ represent the (conjugate) momentum and position of the charge and $\mathbf{r}_m$ the (fixed) position of the dipole. The first term is the kinetic energy of the charge in the presence of the vector potential $\mathbf{A}$ and the second term corresponds to the magnetic dipole interaction at the position of the dipole. The free-photon energy is described by the last term, where $n_{\mathbf{k}\epsilon}$ denotes the number operator for photons with wave vector $\mathbf{k}$ and polarization $\hat{\epsilon}$ and the frequency is related to the wave vector via $\omega = kc$. In terms of photon creation/annihilation operators, the vector potential is expressed in Gaussian units as

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k},\hat{\epsilon}} \sqrt{\frac{2\pi\hbar c^2}{\omega V}}\,\hat{\epsilon}\left(\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}\hat{\epsilon}} + \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} a^{\dagger}_{\mathbf{k}\hat{\epsilon}}\right) \tag{18}$$

where V is the volume of the system. The magnetic field is then given by $\mathbf{B} = \nabla \times \mathbf{A}$.

Suppose that the particle enters the path at time $t = t_i$ and, after circling any number of turns, returns to the starting position at the time $t = t_f$. The state of the particle and the dipole at the initial time is given by $|\Psi; t_i\rangle = |\psi; t_i\rangle|J\rangle$, where $|\psi; t_i\rangle$ denotes the state of the travelling particle and $|J\rangle$ the state of the fixed dipole. Although any number of photons of any momentum and polarization is allowed initially, no additional photon is assumed created or annihilated after the travel because of its low probability. Then the quantum interference between the initial and the final states due to the interaction can be obtained from the element of the time-evolution operator

$$\mathcal{I} \equiv \langle\Psi; t_i|\,\mathrm{Tr}_{\{n_{\mathbf{k}\hat{\epsilon}}\}} \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t\mathcal{H}\right)|\Psi; t_i\rangle = \int \mathrm{d}^3r_f\,\mathrm{d}^3r_i\,\langle\psi; t_i|\mathbf{r}_f\rangle\mathcal{W}(\mathbf{r}_i, \mathbf{r}_f)\langle\mathbf{r}_i|\,\psi; t_i\rangle \tag{19}$$

with

$$\mathcal{W}(\mathbf{r}_i, \mathbf{r}_f) \equiv \langle J|\langle\mathbf{r}_f|\,\mathrm{Tr}_{\{n_{\mathbf{k}\epsilon}\}} \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t\mathcal{H}\right)|\mathbf{r}_i\rangle|J\rangle \tag{20}$$

where the trace is to be performed over all possible photon states. For evaluating the function $\mathcal{W}$, it is natural to use the functional integral representation in terms of the boson coherent state $|\phi\rangle$ together with the position and momentum eigenstates, $|\mathbf{r}\rangle$ and $|\mathbf{p}\rangle$ [20]. The time interval is broken into M time steps of size $\varepsilon \equiv (t_f - t_i)/M$, the closure relation

$$1 = \prod_{\alpha} \frac{\mathrm{d}\phi^*_{\alpha,j}\phi_{\alpha,j}}{2\pi\mathrm{i}}\ \exp\left(-\sum_{\alpha}\phi^*_{\alpha,j}\phi_{\alpha,j}\right)|\phi_{\alpha,j}\rangle\langle\phi_{\alpha,j}| \otimes \mathrm{d}^3r_j|\mathbf{r}_j\rangle\langle\mathbf{r}_j|$$

where α stands for the photon state $(\mathbf{k}, \epsilon)$, is inserted at the jth time step ($j = 1, \ldots, M-1$), and the periodic boundary conditions are used: $\phi_{\alpha,0} = \phi_\alpha$ and $\phi^*_{\alpha,M} = \phi^*_\alpha$ for ϕ, and $\mathbf{r}_0 = \mathbf{r}_i$ and $\mathbf{r}_M = \mathbf{r}_f$ for $\mathbf{r}$. We then obtain

$$\mathcal{W} = \lim_{M\to\infty}\int \prod_{j=1}^{M}\prod_{\alpha} \frac{\mathrm{d}\phi^*_{\alpha,j}\,\mathrm{d}\phi_{\alpha,j}}{2\pi\mathrm{i}} \prod_{j=1}^{M-1}\mathrm{d}^3r_j \prod_{j=1}^{M}\frac{\mathrm{d}^3p_j}{(2\pi\hbar)^3}\,\mathrm{e}^{-\mathcal{S}} \tag{21}$$

where the action is given by

$$\mathcal{S} = \sum_{j=1}^{M}\left\{\sum_{\alpha}[\phi^*_{\alpha,j}\phi_{\alpha,j} - (1 - \mathrm{i}\varepsilon\omega)\phi^*_{\alpha,j}\phi_{\alpha,j-1}] - \frac{\mathrm{i}\varepsilon}{\hbar}\sum_{\alpha}(F^*_{\alpha,j}\phi_{\alpha,j-1} + F_{\alpha,j}\phi^*_{\alpha,j}) - \frac{\mathrm{i}}{\hbar}\mathbf{p}_j\cdot(\mathbf{r}_j - \mathbf{r}_{j-1}) + \frac{\mathrm{i}\varepsilon}{\hbar}\frac{\mathbf{p}_j^2}{2m}\right\} \tag{22}$$

with

$$F_{\alpha,j} \equiv \sqrt{\frac{2\pi\hbar c^2}{\omega V}}\left[\frac{q}{mc}\epsilon\cdot\mathbf{p}_j\ \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_{j-1}) - \mathrm{i}(\mu\cdot\mathbf{k}\times\epsilon)\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_m)\right]. \tag{23}$$

Here we have disregarded the $\mathbf{A}^2$-term describing a two-photon process since its contribution is usually much smaller than the single-photon process.

Since the action in equation (22) is quadratic in the complex variables $\phi_{\alpha,j}$, simple Gaussian integration can be performed over $\phi_{\alpha,j}$, yielding

$$\mathcal{W} = \lim_{M\to\infty} \int \prod_{j=1}^{M-1} \mathrm{d}^3 r_j \prod_{j=1}^{M} \frac{\mathrm{d}^3 p_j}{(2\pi\hbar)^3}\, \mathrm{e}^{-\mathcal{S}} \tag{24}$$

with

$$\mathcal{S} = -\sum_\alpha \frac{\mathrm{i}\varepsilon}{\hbar} \frac{\left|\sum_{j=1}^{M} F_{\alpha,j}\right|^2}{M\hbar\omega} - \sum_{j=1}^{M} \left[\frac{\mathrm{i}}{\hbar} \mathbf{p}_j \cdot (\mathbf{r}_j - \mathbf{r}_{j-1}) - \frac{\mathrm{i}\varepsilon}{\hbar} \frac{\mathbf{p}_j^2}{2m} \right] \tag{25}$$

up to $\mathcal{O}(\varepsilon)$. Here we have omitted a factor coming from the time evolution of photons in which we are not interested. The $\left|\sum F\right|^2$-term contains three contributions: two from the self-energies of the charge and the dipole and one from the charge–dipole interaction. The contribution from the self-energy of the dipole is irrelevant to our analysis and that from the self-energy of the charge is extremely small by several orders of magnitude compared with other terms. Disregarding these two and summing over the photon polarization and momentum by converting $V^{-1}\sum_{\mathbf{k}}$ into $(2\pi)^{-3}\int \mathrm{d}^3 k$, we write the action in the form

$$\mathcal{S} = \sum_{j=1}^{M} \left[\frac{\mathrm{i}\varepsilon}{\hbar} \frac{\mathbf{p}_j^2}{2m} - \frac{\mathrm{i}}{\hbar} \mathbf{p}_j \cdot (\mathbf{r}_j - \mathbf{r}_{j-1}) - \frac{\mathrm{i}\varepsilon}{\hbar} \frac{q}{mc} \frac{\mathbf{p}_j \cdot \mu \times (\mathbf{r}_{j-1} - \mathbf{r}_m)}{|\mathbf{r}_{j-1} - \mathbf{r}_m|^3} \right]. \tag{26}$$

Here the last term may be interpreted as the interaction between the dipole and the magnetic field $\mathbf{B}_q$ generated by the charge in motion:

$$\frac{q}{mc} \frac{\mathbf{p}_j \cdot \mu \times (\mathbf{r}_{j-1} - \mathbf{r}_m)}{|\mathbf{r}_{j-1} - \mathbf{r}_m|^3} = \mu \cdot \mathbf{B}_q(\mathbf{r}_m - \mathbf{r}_{j-1}) \tag{27}$$

where the Biot–Savart law has been used to give $\mathbf{B}_q(\mathbf{r}) = (q/mc)\mathbf{p} \times \mathbf{r}/r^3$. This confirms the interrelation proposed in section 3 between the AB and SAC effects in the consistent quantum-mechanical description.

Tracing out the momentum variables, we obtain the element

$$\mathcal{I} = \lim_{M\to\infty} \frac{1}{\mathcal{N}} \int \prod_{j=0}^{M} \mathrm{d}^3 r_j \langle \psi; t_i | \mathbf{r}_M \rangle\, \mathrm{e}^{-\mathcal{S}} \langle \mathbf{r}_0 | \psi; t_i \rangle \tag{28}$$

with

$$\mathcal{S} = -\frac{\mathrm{i}}{\hbar} \sum_{j=1}^{M} \left[\frac{m}{2\varepsilon} (\mathbf{r}_j - \mathbf{r}_{j-1})^2 + \frac{\varepsilon q^2}{2mc^2} \frac{[\mu \times (\mathbf{r}_{j-1} - \mathbf{r}_m)]^2}{|\mathbf{r}_{j-1} - \mathbf{r}_m|^6} + \frac{q}{c} \frac{(\mathbf{r}_j - \mathbf{r}_{j-1}) \cdot \mu \times (\mathbf{r}_{j-1} - \mathbf{r}_m)}{|\mathbf{r}_{j-1} - \mathbf{r}_m|^3} \right] \tag{29}$$

where $\mathcal{N}$ is an appropriate normalization constant. Without any geometrical assumption, the element $\mathcal{I}$ in equation (28) contains all the contributions coming from any path of the particle. We now take into account only the path shown in figure 3, especially with $\hat{\mathbf{u}} = \hat{\mathbf{z}}$, i.e., $\mu \equiv g\mu_B J\hat{\mathbf{z}}$. In the polar coordinate $\mathbf{r}_j = R(\cos\theta_j, \sin\theta_j, 0)$, we replace the integral over $\mathbf{r}_j$ with one for θ_j, and get

$$\mathcal{I} = \lim_{M\to\infty} \frac{\exp(\mathrm{i}\varphi_1(z_m))}{\mathcal{N}} \int_0^{2\pi} \prod_{j=0}^{M} \frac{\mathrm{d}\theta_j}{2\pi} \langle \psi; t_i | \theta_M \rangle \langle \theta_0 | \psi; t_i \rangle \exp\left(\sum_{j=1}^{M} V(\theta_j - \theta_{j-1}) \right) \tag{30}$$

where we have defined

$$\varphi_1(z_m) \equiv \frac{(t_f - t_i)q^2\mu^2}{2mc^2\,\mathrm{d}_m^4} \qquad \text{and} \qquad V(\theta) \equiv \mathrm{i}\kappa(1-\cos\theta) + \frac{\mathrm{i}\phi_1(z_m)}{2\pi}\sin\theta \tag{31}$$

with $\kappa \equiv mR^2/\hbar\varepsilon$ and

$$\phi_1(z_m) \equiv \frac{2\pi q\mu R^2}{\hbar c d_m^3}. \tag{32}$$

Since $\mathrm{e}^{V(\theta)}$ is periodic in θ, it can be expressed as a Fourier series: $\mathrm{e}^{V(\theta)} = \sum_{s=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}s\theta}\,\mathrm{e}^{\widetilde{V}(s)}$, where $\mathrm{e}^{\widetilde{V}(s)} = \int_{-\pi}^{\pi}(\mathrm{d}\theta/2\pi)\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{e}^{V(\theta)}$. Using this Fourier expansion of $\exp(V(\theta_j - \theta_{j-1}))$ and integrating over θ_j for $1 \leqslant j \leqslant M-1$, we obtain

$$\mathcal{I} = \lim_{M\to\infty} \frac{\exp(\mathrm{i}\varphi_1(z_m))}{\mathcal{N}} \int_0^{2\pi} \frac{\mathrm{d}\theta_0\,\mathrm{d}\theta_M}{(2\pi)^2} \langle\psi; t_i|\theta_M\rangle\langle\theta_0|\psi; t_i\rangle \times \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\xi\, \exp(M\widetilde{V}(\xi))\exp(\mathrm{i}n(\theta_M - \theta_0) + 2\pi\mathrm{i}n\xi) \tag{33}$$

where we have used the fact that the integral $\int_0^{2\pi}(\mathrm{d}\theta/2\pi)\,\mathrm{e}^{\mathrm{i}s\theta}$ gives rise to the Kronecker delta function $\delta_{s,0}$ and applied the Poisson summation formula to the summation variable.

Let us consider the case where the charge enters initially into the circle at $\theta = 0$, such that the initial wavefunction is given by the delta function: $\langle\theta|\psi; t_i\rangle = 2\pi\delta(\theta)$. Then θ_0 and θ_M are set equal to zero and we have, from the definition of $\widetilde{V}(\xi)$,

$$\int_{-\infty}^{\infty} \mathrm{d}\xi\, \exp(M\widetilde{V}(\xi) + 2\pi\mathrm{i}n\xi) = \int_{-\infty}^{\infty} \mathrm{d}\xi \left[\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \exp(-\mathrm{i}\xi\theta)\exp(\mathrm{i}(\kappa(1-\cos\theta) + (\phi_1/2\pi)\sin\theta))\right]^M \exp(2\pi\mathrm{i}n\xi)$$

where in the limit $M \to \infty$ or $\kappa \to \infty$, the integrand except near $\theta = 0$ oscillates rapidly to be inconsequential so that we may approximate $\sin\theta \approx \theta$. Shifting the variable ξ by $\phi_1(z_m)/2\pi$, we obtain

$$\mathcal{I} = \exp(\mathrm{i}\varphi_1(z_m)) \sum_n \exp(\mathrm{i}n\phi_1(z_m))\mathcal{I}_0(n) \tag{34}$$

where n represents the winding number of the charge around the dipole and $\mathcal{I}_0(n)$ is the interference amplitude for a free particle (with the winding number n) in the absence of the magnetic field. The phase shift by the photon-mediated interaction between the charge and the dipole thus consists of the winding-number dependent part and the time-dependent (dynamical) one: $n\phi_1(z_m) + \varphi_1(z_m)$. Note that the former as well as the latter still depend on the size and the relative position of the path, and are not topological at all.

We now investigate the original AB setup shown in figure 4. We assume that the magnet has an infinitely long cylindrical structure with radius R_m ($<R$) and is composed of N identical atoms or molecules per unit volume, each of which has the same dipole moment given before. Then the dipole interaction term $-\mu\cdot\mathbf{B}(\mathbf{r}_m)$ in the Hamiltonian (17) is replaced by $-\sum_{\mathbf{r}_m}\mu\cdot\mathbf{B}(\mathbf{r}_m)$. Taking the z-axis along the cylindrical axis of the magnet, namely $\hat{\mathbf{u}} = \hat{\mathbf{z}}$, we follow the same procedure as that for a single dipole to reach equation (26) with the additional summation over $\mathbf{r}_m$ in the third term; the self-energy contribution of the magnet is also modified but it is irrelevant as before. Converting the summation $\sum_{\mathbf{r}_m}$ into the integral

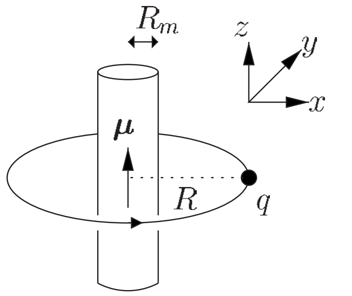

Figure 4. A particle with charge q circles around an infinitely long cylindrical magnet along the z-direction. As before, the trajectory of the particle is confined to the xy-plane.

$N\int_{\mathbf{r}_m} \mathrm{d}^3\mathbf{r}_m$ extended to the volume of the magnet and performing the integration, we obtain, in the place of equation (26),

$$\mathcal{S} = \sum_{j=1}^{M}\left[\frac{\mathrm{i}\varepsilon}{\hbar}\frac{\mathbf{p}_j^2}{2m} - \frac{\mathrm{i}}{\hbar}\mathbf{p}_j\cdot(\mathbf{r}_j - \mathbf{r}_{j-1}) + \frac{\mathrm{i}\varepsilon}{\hbar}\frac{q\Phi}{2\pi mc}\frac{\mathbf{p}_j\cdot\mathbf{z}\times\mathbf{r}_{j-1}}{\mathbf{r}_{j-1}^2}\right] \tag{35}$$

where $\Phi \equiv (4\pi N\mu)\left(\pi R_m^2\right)$ is the magnetic flux inside the magnet. Following the remaining procedure leads to equation (34) with the replacement of ϕ_1 and φ_1 by

$$\phi_{\mathrm{AB}} \equiv \frac{q\Phi}{\hbar c} \qquad \text{and} \qquad \varphi \equiv \frac{(t_f - t_i)\hbar\phi_{\mathrm{AB}}^2}{8\pi^2 m R^2} \tag{36}$$

respectively. Referring to the data used in an AB interferometry experiment [17], we observe that the dynamical part φ is of the order of $\lambda_{\mathrm{deB}}/2\pi R$, where λ_{deB} is the de Broglie wavelength of the charge, and negligibly small by several orders of magnitude in comparison with ϕ_{AB}. On the other hand, for winding number n, the topological phase shift is given by $n\phi_{\mathrm{AB}}$, which is the same as that in the conventional AB effect.

This result can also be obtained from the phase shift generated by a single dipole in equation (32) in the following way: consider a charged particle circling around an infinitely long wire along the z-direction, with the linear density N of magnetic dipoles. The total phase shift for winding number n is then given by

$$\sum_{\mathbf{r}_m} n\phi(z_m) = N\int_{-\infty}^{\infty}\mathrm{d}z_m n\phi(z_m) = \frac{nq(4\pi\mu N)}{\hbar c}. \tag{37}$$

Identifying the magnetic flux $\Phi = 4\pi\mu N$ here, we obtain the same topological phase $n\phi_{\mathrm{AB}}$, which manifests that the topological effect arises from the non-simply-connected geometry of the space.

5. Precession of magnetic dipoles

In previous sections, we have assumed that magnetic dipoles are inert and retain their ground state throughout the experiment, regardless of their interactions with electromagnetic fields or potentials. Such inertness of magnetic dipoles is a crucial ingredient in acquiring well-defined topological phases and interrelations between the AB and AC effects. Fortunately, this condition can usually be justified in real experimental situations. In AB/SAB experiment, the dipoles in the (cylindrical) magnet have very strong ferromagnetic interactions between them, giving rise to negligibly small transition probability to excited states by the interaction with the magnetic field generated by an electric charge in motion [13]. Similarly, the neutron used as a neutral particle with non-zero magnetic moment in AC/SAC experiment hardly excites to higher-energy states, with the energy level several hundred MeV above the ground state, by the very weak interaction with the magnetic field of charges [15]. For this reason, the magnetic dipole may be regarded as an inert object whose state is not affected by the field of the charge in relative motion, and the adiabatic condition is satisfied.

Then what if the proviso for the inertness is relaxed? It has been shown that, in the opposite limit where the magnetic dipole is replaced by a classical object such as a thin circular pipe containing a charged fluid, the phase shift for the AB and SAC effects disappears completely [13, 15]. The reason is that the kinetic energy of charges in the classical object changes in the presence of an external magnetic field in such a way that it cancels out exactly the contribution of the magnetic field to the action, making the action independent of the field.

More profoundly, in the absence of the adiabatic condition, one cannot replace the matrix σ_3 in equation (9) by its eigenvalue and has to treat the spin as a dynamic quantum variable. As shown in section 3, the magnetic moment always couples to the magnetic field induced by the charge in relative motion in both the AB/SAB and the AC/SAC effects. Consequently, the magnetic moment precesses around the magnetic field and the accumulation of the quantum phase shift is accompanied by the local precession. There has been an argument against the nonlocality of the AC/SAC effect, based on the commutativity and the linear relation between the phase shift and the precession angle which can be locally observable [6]. If we follow their assertion, however, the interrelation demonstrated in section 3 leads us to conclude that the AB/SAB effect is also local. Recently, flaws in their argument have been pointed out [7, 8]: since the phase shift and the precession angle are noncanonical variables, the commutativity between them does not necessarily imply mutual observability [21] and instead the observation of the local precession induces uncertainty in the phase shift. (Here a noncanonical variable means an observable depending on the Hamiltonian of the system.) Therefore even if the magnetic dipole can precess, it should not hurt the nonlocal and topological nature of the AB and AC effects.

However, it can give a quantitative correction in the phase shift acquired since the precession of the dipole due to the motion of the charge in turn affects the state of the charge. Let us consider the AB effect microscopically as in section 4 and now suppose that the dipole can precess or be excited energetically as it interacts via photons. Then the entire system will culminate in some entangled state of the charge and the dipole just before the charge interferes with its initial state or in other words it hits the screen in the AB interferometry. With respect to each final state of the dipole, the charge may acquire different phases and the states of different phases may interfere with each other to obscure the interference pattern on the screen. The correction in the interference, however, may be small compared with the topological phase because it involves at least two-photon processes, one from the charge to the dipole and one in return; the main contribution to the phase shift comes from the one-photon process from the dipole to the charge. Hence we may neglect the effects of the precession on the charge interference pattern as long as the precession of the dipole moment is not too large, which is expected to be well satisfied in experiment.

6. Conclusions

In this work, we have established systematic unification of the AB and AC effects together with their scalar counterparts, making clear two kinds of correspondence: formal and physical. There exist formal equivalences between the AB and AC effects and between the SAB and SAC effects. They are based on the multiply-connected geometry of the space, where the potential or the field can be gauged away so that their effects can be absorbed into the phase of the wavefunction, together with the adiabatic condition for the quantum state of magnetic dipoles. The list of electromagnetic conditions for these four effects indicates their completeness in topological phases which can be acquired by systems of electric charges and magnetic dipoles. In addition, by devising the structure of the source for the SAB effect which has been missing, we have also completed the analysis of the two-body interaction and found that the same two-body interactions between an electric charge and a magnetic dipole are responsible for the AB and the SAC effects and for the AC and SAB effects, which suggests the physical interrelations between them.

The importance of quantum treatment of the total system encompassing the source has been appreciated during the consolidating analysis. Accordingly, for precise quantum results, we have given an exact description of the AB effect by means of the path integral representation. Beginning with a system consisting of an electric charge and a magnetic dipole, interacting with each other via virtual photons, we have confirmed the result of the conventional approach and manifested the topological nature of the AB effect. Finally, the adiabatic condition has been argued to hold in real experiments and the survival of the nonlocality of the topological phase been discussed in the presence of the precession of the magnetic dipole.

Recently, by exploiting the electromagnetic duality in the Maxwell equations, some authors have suggested new kinds of gedanken topological effects for a magnetic monopole or an electric dipole [14, 16]. Since their equations of motion have exactly the same structure as those of their dual effects, i.e., the AB and AC effects, all the relations between the effects found in this paper can be directly applied to them simply by substituting magnetic charges and electric dipoles for electric charges and magnetic dipoles. In addition, following our argument, one can find the structure of their scalar counterparts systematically [7]. Hence our results encompass essentially all the topological effects implied in quantum electromagnetic theory.

Acknowledgments

We thank M-S Choi for useful discussions and acknowledge the partial support from the SKOREA Program and from the BK21 Program.

References

[1] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
[2] Aharonov Y and Casher A 1984 *Phys. Rev. Lett.* **53** 319
[3] Zeilinger A 1985 *Fundamental Aspects of Quantum Theory (NATO Advanced Study Institute Series B vol 144)* ed V Gorini and A Frigerio (New York: Plenum) p 311
Anandan J 1989 *Phys. Lett.* A **138** 347
Anandan J 1991 *Phys. Lett.* **152** 504
[4] Boyer T H 1987 *Phys. Rev.* A **36** 5083
[5] Aharonov Y, Pearle P and Vaidman L 1988 *Phys. Rev.* A **37** 4052
[6] Peshkin M and Lipkin H J 1995 *Phys. Rev. Lett.* **74** 2847
Hyllus P and Sjöqvist E 2002 *Phys. Rev. Lett.* **89** 198901
[7] Aharonov Y and Reznik B 2000 *Phys. Rev. Lett.* **84** 4790
[8] Aharonov Y and Reznik B 2002 *Phys. Rev. Lett.* **89** 198902
[9] Hagen C R 1990 *Phys. Rev. Lett.* **64** 2347
[10] He X G and McKellar B H J 1991 *Phys. Lett.* B **256** 250
[11] Goldhaber A S 1989 *Phys. Rev. Lett.* **62** 482
[12] Santos E and Gonzalo I 1999 *Europhys. Lett.* **45** 418

[13] Comay E 2000 *Phys. Rev.* A **62** 042102
[14] Dowling J P, Williams C P and Franson J D 1999 *Phys. Rev. Lett.* **83** 2486
[15] Comay E 1998 *Phys. Lett.* A **250** 12
[16] He X G and McKellar B H J 1993 *Phys. Rev.* A **47** 3424
Wilkens M 1994 *Phys. Rev. Lett.* **72** 5
Yi J, Jeon G S and Choi M Y 1995 *Phys. Rev.* B **52** 7838
Lee T-Y 2000 *Phys. Rev.* A **62** 064101
[17] Tonomura A *et al* 1986 *Phys. Rev. Lett.* **56** 792 and earlier references cited therein
[18] Cimmino A *et al* 1989 *Phys. Rev. Lett.* **63** 380
Sangster K, Hinds E A, Barnett S M and Riis E 1993 *Phys. Rev. Lett.* **71** 3641
[19] Allman B E *et al* 1992 *Phys. Rev. Lett.* **68** 2409
Allman B E, Lee W-T, Motrunich O I and Werner S A 1999 *Phys. Rev.* A **60** 4272
[20] See, e.g., Negele J W and Orland H 1998 *Quantum Many-Particle Systems* (Redwood City, CA: Addison-Wesley)
[21] Aharonov Y and Safko J L 1975 *Ann. Phys., NY* **91** 279